MA114 Summer 2018
Worksheet 12 – Root and Ratio Tests – 7/02/18

1. For each series below, apply the ratio or root test to determine the convergence of the series. If the test is inconclusive, say so.

(or ratio test)

(a) $\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$ $L = \lim_{n\to\infty} \left(\frac{10^n}{2^{n^2}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{10}{2^n}\right) = 0,$

Since $L = 0 < 1$, $\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$ converges by Root Test.

(b) $\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$ $L = \lim_{k\to\infty} \left(\left(\frac{k}{k+10}\right)^k\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+10} = 1.$

Since $L = 1$, the Root Test is inconclusive:

Try Div Test: $\lim_{k\to\infty} \left(\frac{k}{k+10}\right)^k$ is hard, so let's find $\lim_{k\to\infty} \ln\left(\left(\frac{k}{k+10}\right)^k\right) = \lim_{k\to\infty} k \ln\left(\frac{k}{k+10}\right) = \cancel{\lim_{k\to\infty} \frac{\ln\left(\frac{k}{k+10}\right)}{\frac{1}{k}} = \frac{\infty}{\infty}}$ $= \infty \cdot \ln(1) = \infty \cdot 0$ (rewrite to use L'H)

$$= \lim_{k\to\infty} \frac{\ln\left(\frac{k}{k+10}\right)}{\frac{1}{k}} \stackrel{L'H}{=} \lim_{k\to\infty} \frac{\frac{1}{\frac{k}{k+10}} \cdot \frac{k+10-k}{(k+10)^2}}{-\frac{1}{k^2}} = -\frac{10k^2}{k(k+10)} = -10.$$

So $\left(\frac{k}{k+10}\right)^k \to e^{-10}$ as $k\to\infty$.

Thus the series diverges since $\lim_{k\to\infty} a_k = e^{-10} \neq 0$.

(c) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ Use Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \lim_{n\to\infty} e^{n+1-n} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty} \frac{e}{n+1} = 0.$$

Since $L = 0 < 1$, the series converges absolutely by the Ratio Test.

(d) $\sum_{m=2}^{\infty} \frac{1}{5^m + 1}$ Use Root Test (or ratio):

$$L = \lim_{m\to\infty} \sqrt[m]{|a_m|} = \lim_{m\to\infty} \left(\frac{1}{5^m+1}\right)^{\frac{1}{m}} = \lim_{m\to\infty} \left(\frac{1}{5^m\left(1+\left(\frac{1}{5}\right)^m\right)}\right)^{\frac{1}{m}} = \lim_{m\to\infty} \frac{1}{5\left(1+\frac{1}{5^m}\right)^{\frac{1}{m}}} = \frac{1}{5(1+0)^0} = \frac{1}{5}.$$

Since $L = \frac{1}{5} < 1$, the series converges absolutely by the Root Test.